

# Topic 8: Functions of Several Variables and Partial Differentiation

(from Stewart Calculus)

## 8.1 FUNCTIONS OF TWO VARIABLES

### 8.1.1 Definition

**DEFINITION** A function  $f$  of two variables is a rule that assigns to each ordered pair of real numbers  $(x, y)$  in a set  $D$  a unique real number denoted by  $f(x, y)$ . The set  $D$  is the **domain** of  $f$  and its **range** is the set of values that  $f$  takes on, that is,  $\{f(x, y) \mid (x, y) \in D\}$ .

Normally written as  $z = f(x, y)$  where

$x$  and  $y$  are the independent variables and  $z$  is the dependent variable.

Domain -- subset of  $\mathfrak{R}^2$

Range -- subset of  $\mathfrak{R}$

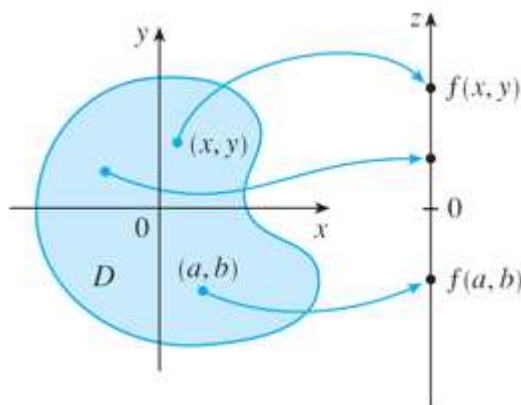


Figure 8.1

**EXAMPLE** For each of the following functions, evaluate  $f(3, 2)$  and find the domain.

(a)  $f(x, y) = \frac{\sqrt{x + y + 1}}{x - 1}$

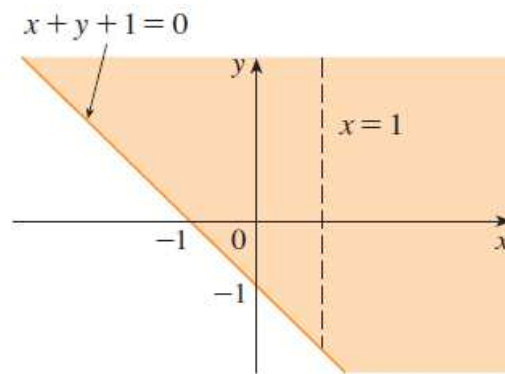
(b)  $f(x, y) = x \ln(y^2 - x)$

**SOLUTION**

(a) 
$$f(3, 2) = \frac{\sqrt{3 + 2 + 1}}{3 - 1} = \frac{\sqrt{6}}{2}$$

The expression for  $f$  makes sense if the denominator is not 0 and the quantity under the square root sign is nonnegative. So the domain of  $f$  is

$$D = \{(x, y) \mid x + y + 1 \geq 0, x \neq 1\}$$

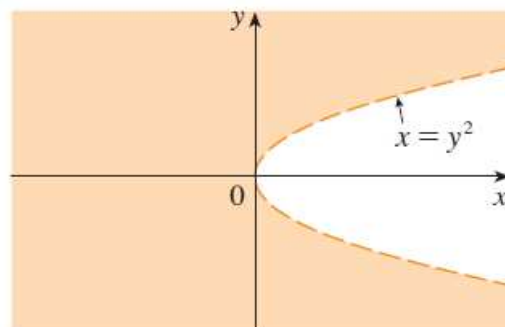


$$\text{Domain of } f(x, y) = \frac{\sqrt{x+y+1}}{x-1}$$

Figure 8.2

$$(b) \quad f(3, 2) = 3 \ln(2^2 - 3) = 3 \ln 1 = 0$$

Since  $\ln(y^2 - x)$  is defined only when  $y^2 - x > 0$ , that is,  $x < y^2$ , the domain of  $f$  is  $D = \{(x, y) \mid x < y^2\}$ . This is the set of points to the left of the parabola  $x = y^2$ .



$$\text{Domain of } f(x, y) = x \ln(y^2 - x)$$

Figure 8.3

**EXAMPLE** Find the domain and range of  $g(x, y) = \sqrt{9 - x^2 - y^2}$ .

**SOLUTION** The domain of  $g$  is

$$D = \{(x, y) \mid 9 - x^2 - y^2 \geq 0\} = \{(x, y) \mid x^2 + y^2 \leq 9\}$$

which is the disk with center  $(0, 0)$  and radius 3. (See Figure 4.) The range of  $g$  is

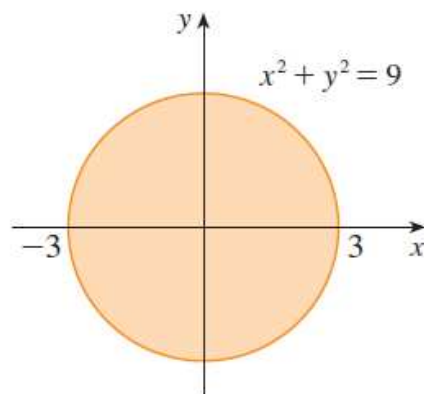
$$\{z \mid z = \sqrt{9 - x^2 - y^2}, (x, y) \in D\}$$

Since  $z$  is a positive square root,  $z \geq 0$ . Also

$$9 - x^2 - y^2 \leq 9 \Rightarrow \sqrt{9 - x^2 - y^2} \leq 3$$

So the range is

$$\{z \mid 0 \leq z \leq 3\} = [0, 3]$$

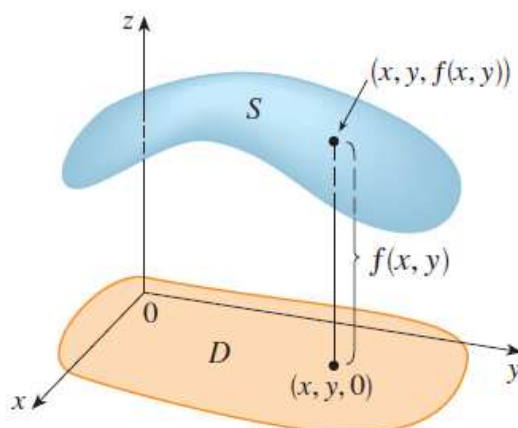


$$\text{Domain of } g(x, y) = \sqrt{9 - x^2 - y^2}$$

Figure 8.4

### 8.1.2 Definition using Graph

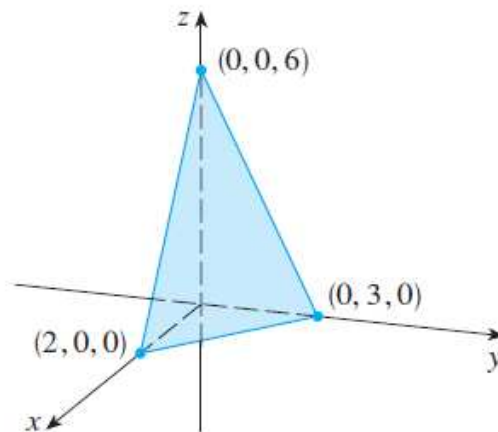
**DEFINITION** If  $f$  is a function of two variables with domain  $D$ , then the **graph** of  $f$  is the set of all points  $(x, y, z)$  in  $\mathbb{R}^3$  such that  $z = f(x, y)$  and  $(x, y)$  is in  $D$ .



Just as the graph of a function  $f$  of one variable is a curve  $C$  with equation  $y = f(x)$ , so the graph of a function  $f$  of two variables is a surface  $S$  with equation  $z = f(x, y)$ . We can visualize the graph  $S$  of  $f$  as lying directly above or below its domain  $D$  in the  $xy$ -plane.

**EXAMPLE** Sketch the graph of the function  $f(x, y) = 6 - 3x - 2y$ .

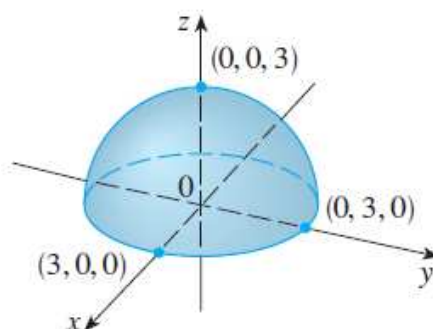
**SOLUTION** The graph of  $f$  has the equation  $z = 6 - 3x - 2y$ , or  $3x + 2y + z = 6$ , which represents a plane. To graph the plane we first find the intercepts. Putting  $y = z = 0$  in the equation, we get  $x = 2$  as the  $x$ -intercept. Similarly, the  $y$ -intercept is 3 and the  $z$ -intercept is 6. This helps us sketch the portion of the graph that lies in the first octant.



**Figure 8.5**

**EXAMPLE** Sketch the graph of  $g(x, y) = \sqrt{9 - x^2 - y^2}$ .

**SOLUTION** The graph has equation  $z = \sqrt{9 - x^2 - y^2}$ . We square both sides of this equation to obtain  $z^2 = 9 - x^2 - y^2$ , or  $x^2 + y^2 + z^2 = 9$ , which we recognize as an equation of the sphere with center the origin and radius 3. But, since  $z \geq 0$ , the graph of  $g$  is just the top half of this sphere



Graph of  $g(x, y) = \sqrt{9 - x^2 - y^2}$

**Figure 8.6**

## 8.2 PARTIAL DERIVATIVES

In general, if  $f$  is a function of two variables  $x$  and  $y$ , suppose we let only  $x$  vary while keeping  $y$  fixed, say  $y = b$ , where  $b$  is a constant. Then we are really considering a function of a single variable  $x$ , namely,  $g(x) = f(x, b)$ . If  $g$  has a derivative at  $a$ , then we call it the **partial derivative of  $f$  with respect to  $x$  at  $(a, b)$**  and denote it by  $f_x(a, b)$ . Thus

$$\boxed{1} \quad f_x(a, b) = g'(a) \quad \text{where} \quad g(x) = f(x, b)$$

By the definition of a derivative, we have

$$g'(a) = \lim_{h \rightarrow 0} \frac{g(a+h) - g(a)}{h}$$

and so Equation 1 becomes

$$\boxed{2} \quad f_x(a, b) = \lim_{h \rightarrow 0} \frac{f(a+h, b) - f(a, b)}{h}$$

Similarly, the **partial derivative of  $f$  with respect to  $y$  at  $(a, b)$** , denoted by  $f_y(a, b)$ , is obtained by keeping  $x$  fixed ( $x = a$ ) and finding the ordinary derivative at  $b$  of the function  $G(y) = f(a, y)$ :

$$\boxed{3} \quad f_y(a, b) = \lim_{h \rightarrow 0} \frac{f(a, b+h) - f(a, b)}{h}$$

If we now let the point  $(a, b)$  vary in Equations 2 and 3,  $f_x$  and  $f_y$  become functions of two variables.

**4** If  $f$  is a function of two variables, its **partial derivatives** are the functions  $f_x$  and  $f_y$  defined by

$$f_x(x, y) = \lim_{h \rightarrow 0} \frac{f(x+h, y) - f(x, y)}{h}$$

$$f_y(x, y) = \lim_{h \rightarrow 0} \frac{f(x, y+h) - f(x, y)}{h}$$



**NOTATIONS FOR PARTIAL DERIVATIVES** If  $z = f(x, y)$ , we write

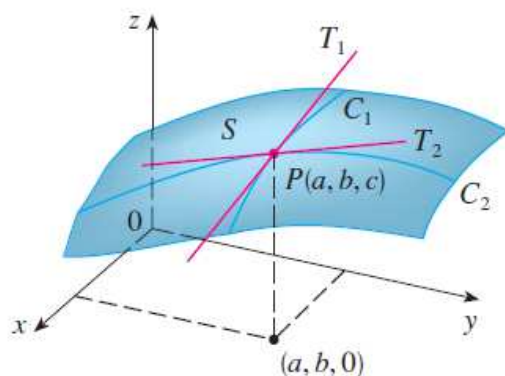
$$f_x(x, y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x, y) = \frac{\partial z}{\partial x} = f_1 = D_1 f = D_x f$$

$$f_y(x, y) = f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} f(x, y) = \frac{\partial z}{\partial y} = f_2 = D_2 f = D_y f$$

**RULE FOR FINDING PARTIAL DERIVATIVES OF  $z = f(x, y)$**

1. To find  $f_x$ , regard  $y$  as a constant and differentiate  $f(x, y)$  with respect to  $x$ .
2. To find  $f_y$ , regard  $x$  as a constant and differentiate  $f(x, y)$  with respect to  $y$ .

**In diagram**



The partial derivatives of  $f$  at  $(a, b)$  are the slopes of the tangents to  $C_1$  and  $C_2$ .

**Figure 8.7**

**EXAMPLE** If  $f(x, y) = x^3 + x^2y^3 - 2y^2$ , find  $f_x(2, 1)$  and  $f_y(2, 1)$ .

**SOLUTION** Holding  $y$  constant and differentiating with respect to  $x$ , we get

$$f_x(x, y) = 3x^2 + 2xy^3$$

and so

$$f_x(2, 1) = 3 \cdot 2^2 + 2 \cdot 2 \cdot 1^3 = 16$$

Holding  $x$  constant and differentiating with respect to  $y$ , we get

$$f_y(x, y) = 3x^2y^2 - 4y$$

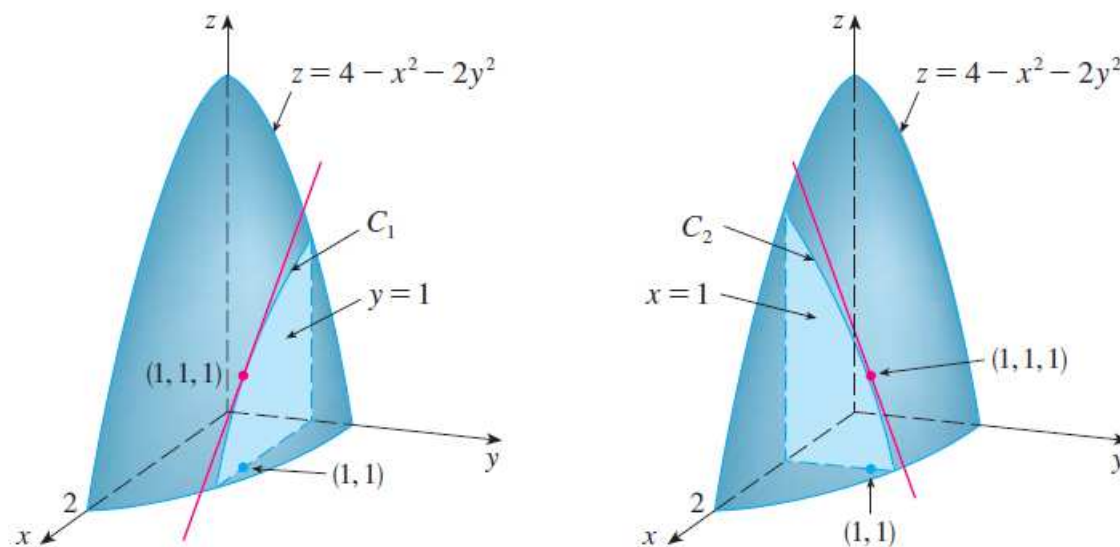
$$f_y(2, 1) = 3 \cdot 2^2 \cdot 1^2 - 4 \cdot 1 = 8$$

**EXAMPLE** If  $f(x, y) = 4 - x^2 - 2y^2$ , find  $f_x(1, 1)$  and  $f_y(1, 1)$  and interpret these numbers as slopes.

**SOLUTION** We have

$$f_x(x, y) = -2x \quad f_y(x, y) = -4y$$

$$f_x(1, 1) = -2 \quad f_y(1, 1) = -4$$



**Figure 8.8**

**EXAMPLE** If  $f(x, y) = \sin\left(\frac{x}{1+y}\right)$ , calculate  $\frac{\partial f}{\partial x}$  and  $\frac{\partial f}{\partial y}$ .

**SOLUTION** Using the Chain Rule for functions of one variable, we have

$$\frac{\partial f}{\partial x} = \cos\left(\frac{x}{1+y}\right) \cdot \frac{\partial}{\partial x} \left(\frac{x}{1+y}\right) = \cos\left(\frac{x}{1+y}\right) \cdot \frac{1}{1+y}$$

$$\frac{\partial f}{\partial y} = \cos\left(\frac{x}{1+y}\right) \cdot \frac{\partial}{\partial y} \left(\frac{x}{1+y}\right) = -\cos\left(\frac{x}{1+y}\right) \cdot \frac{x}{(1+y)^2}$$

### 8.3 CHAIN RULE

Recall that the Chain Rule for functions of a single variable gives the rule for differentiating a composite function: If  $y = f(x)$  and  $x = g(t)$ , where  $f$  and  $g$  are differentiable functions, then  $y$  is indirectly a differentiable function of  $t$  and

$$\boxed{1} \quad \frac{dy}{dt} = \frac{dy}{dx} \frac{dx}{dt}$$

**2 THE CHAIN RULE (CASE I)** Suppose that  $z = f(x, y)$  is a differentiable function of  $x$  and  $y$ , where  $x = g(t)$  and  $y = h(t)$  are both differentiable functions of  $t$ . Then  $z$  is a differentiable function of  $t$  and

$$\frac{dz}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt}$$

**EXAMPLE** If  $z = x^2y + 3xy^4$ , where  $x = \sin 2t$  and  $y = \cos t$ , find  $dz/dt$  when  $t = 0$ .

**SOLUTION** The Chain Rule gives

$$\begin{aligned} \frac{dz}{dt} &= \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt} \\ &= (2xy + 3y^4)(2 \cos 2t) + (x^2 + 12xy^3)(-\sin t) \end{aligned}$$

It's not necessary to substitute the expressions for  $x$  and  $y$  in terms of  $t$ . We simply observe that when  $t = 0$ , we have  $x = \sin 0 = 0$  and  $y = \cos 0 = 1$ . Therefore

$$\left. \frac{dz}{dt} \right|_{t=0} = (0 + 3)(2 \cos 0) + (0 + 0)(-\sin 0) = 6$$

**EXAMPLE** The pressure  $P$  (in kilopascals), volume  $V$  (in liters), and temperature  $T$  (in kelvins) of a mole of an ideal gas are related by the equation  $PV = 8.31T$ . Find the rate at which the pressure is changing when the temperature is 300 K and increasing at a rate of 0.1 K/s and the volume is 100 L and increasing at a rate of 0.2 L/s.

**SOLUTION** If  $t$  represents the time elapsed in seconds, then at the given instant we have  $T = 300$ ,  $dT/dt = 0.1$ ,  $V = 100$ ,  $dV/dt = 0.2$ . Since

$$P = 8.31 \frac{T}{V}$$



the Chain Rule gives

$$\begin{aligned}\frac{dP}{dt} &= \frac{\partial P}{\partial T} \frac{dT}{dt} + \frac{\partial P}{\partial V} \frac{dV}{dt} = \frac{8.31}{V} \frac{dT}{dt} - \frac{8.31T}{V^2} \frac{dV}{dt} \\ &= \frac{8.31}{100} (0.1) - \frac{8.31(300)}{100^2} (0.2) = -0.04155\end{aligned}$$

The pressure is decreasing at a rate of about 0.042 kPa/s.

**3 THE CHAIN RULE (CASE 2)** Suppose that  $z = f(x, y)$  is a differentiable function of  $x$  and  $y$ , where  $x = g(s, t)$  and  $y = h(s, t)$  are differentiable functions of  $s$  and  $t$ . Then

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial s} \qquad \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial t}$$

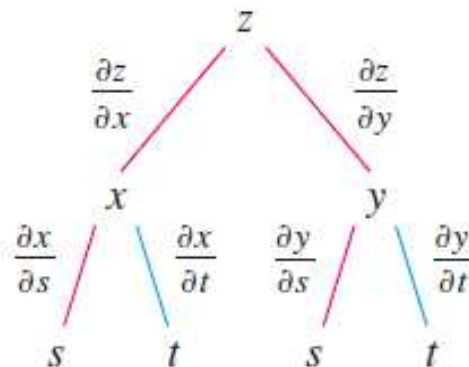
**EXAMPLE** If  $z = e^x \sin y$ , where  $x = st^2$  and  $y = s^2t$ , find  $\partial z/\partial s$  and  $\partial z/\partial t$ .

**SOLUTION** Applying Case 2 of the Chain Rule, we get

$$\begin{aligned}\frac{\partial z}{\partial s} &= \frac{\partial z}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial s} = (e^x \sin y)(t^2) + (e^x \cos y)(2st) \\ &= t^2 e^{st^2} \sin(s^2t) + 2ste^{st^2} \cos(s^2t)\end{aligned}$$

$$\begin{aligned}\frac{\partial z}{\partial t} &= \frac{\partial z}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial t} = (e^x \sin y)(2st) + (e^x \cos y)(s^2) \\ &= 2ste^{st^2} \sin(s^2t) + s^2 e^{st^2} \cos(s^2t)\end{aligned}$$

Case 2 of the Chain Rule contains three types of variables:  $s$  and  $t$  are **independent** variables,  $x$  and  $y$  are called **intermediate** variables, and  $z$  is the **dependent** variable. Notice that Theorem 3 has one term for each intermediate variable and each of these terms resembles the one-dimensional Chain Rule in Equation 1.



**Figure 8.9** Tree diagram